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SEMILATTICES OF TWO CLASSES OF SEMIGROUPS

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Abstract

Right congruences $\mathcal{J}^{**}$ and $\mathcal{H}^{**}$ on a semigroup are introduced, and semigroups which are semilattices of $\mathcal{J}^{**}$-simple monoids or $\mathcal{H}^{**}$-simple monoids are characterized.

1. Introduction

For standard terminology and notation in semigroup theory, see Howie [3].

Green [2] introduced a $\mathcal{L}$-relation on any semigroup $S : a, b \in S$ are $\mathcal{L}$-related if and only if a and b generate the same principal left ideal.

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For $a, b \in S$, let $a\mathcal{L}^*b$ if and only if a and b are $\mathcal{L}$-related in some over semigroup of S. It was shown that $a\mathcal{L}^*b$ if and only if for all $x, y \in S^1$,

$$ax = ay \text{ if and only if } bx = by$$

(see Fountain [1]).

For any equivalence relation σ on a semigroup S, let σ^r be the relation on S defined by: for any $a, b \in S$, $a\sigma^r b$ if and only if for all $x, y \in S^1$,

$$ax\,\sigma\, ay \text{ if and only if } bx\,\sigma\, by.$$

It is easy to see that σ^r is a right congruence on S.

In particular when $\sigma = 1_S$, the equality on S, then $\mathcal{L}^* = 1_S^r$. A particular case is the case where $\sigma = \mathcal{R}$, which was discussed in [5]. The notation $\mathcal{L}^{**} = \mathcal{R}^r$ was introduced and semigroups which are semilattices of $\mathcal{L}^{**}$-simple monoids was characterized in [5]. Another particular case is the case where $\sigma = \mathcal{D}$, which was discussed in [6]. The notation $\mathcal{D}^{**} = \mathcal{D}^r$ was introduced and semigroups which are semilattices of $\mathcal{D}^{**}$-simple monoids was characterized in [6]. We known that Green's equivalences are five ones $\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{J}$, and $\mathcal{L}$ and $\mathcal{R}$ are dual. The purpose of this paper is to introduce right congruences $\mathcal{J}^r$ and $\mathcal{H}^r$, denoted by $\mathcal{J}^{**}$ and $\mathcal{H}^{**}$, respectively, and to give structure theorems for the class of all semigroups where all idempotents are central and each $\mathcal{J}^{**}$-class or $\mathcal{H}^{**}$-class contains an idempotent.

2. Semilattices of $\mathcal{K}^{**}$-Simple Semigroups

A particular case is the case where $\sigma = \mathcal{J}$, and another particular case is the case where $\sigma = \mathcal{H}$, which are discussed in this section. Let $\mathcal{K} \in \{\mathcal{J}, \mathcal{H}\}$. The notation $\mathcal{K}^{**} = \mathcal{K}^{r}$ is introduced. The $\mathcal{K}^{**}$-class containing the element a is denoted by K_a^{**}. As usual, $E(S)$ is the set of idempotents of a semigroup S.

As the proof of Lemma 1 in [6], we can obtain the following:

Lemma 1. *Let S be a semigroup. If the $\mathcal{K}^{**}$-class K_a^{**} contains an idempotent e, then $a\mathcal{K}ae$.*

Lemma 2 ([5])**.** *Let S be a semigroup in which the idempotents are central. If σ is a right congruence on S such that each σ-class contains a unique idempotent, then σ is a semilattice congruence on S, and $S / \sigma \cong E(S)$.*

Lemma 3. *Let S be a semigroup in which the idempotents are central. If each $\mathcal{K}^{**}$-class contains an idempotent, then $\mathcal{K}^{**}$ is a semilattice congruence on S, $S / \mathcal{K}^{**} \cong E(S)$, and each $\mathcal{K}^{**}$-class is a monoid.*

Proof. (1) When $\mathcal{K} = \mathcal{J}$. By Lemma 1, if the $\mathcal{J}^{**}$-class J_a^{**} contains an idempotent e, then $a\mathcal{J}ae$. Hence there exists $x, y \in S^1$ such that $a = xaey$. Since e is in the central of S, we have that $xae = exa$, hence $a = exay$. Therefore $ea = a = ae$. Let $b, c \in J_a^{**} = J_e^{**}$. Then $bc\mathcal{J}^{**}ec$ since $\mathcal{J}^{**}$ is a right congruence on S. Now $ec = c$, we have $bc\mathcal{J}^{**}c\mathcal{J}^{**}a$. We obtain that J_a^{**} is a monoid with identity e. Since the identity of a monoid is unique, using the centrality of the idempotent, we have that each $\mathcal{J}^{**}$-class contains a unique idempotent.

Since $\mathcal{J}^{**}$ is a right congruence on S, it follows from Lemma 2 that $\mathcal{J}^{**}$ is a semilattice congruence on S, and $S / \mathcal{J}^{**} \cong E(S)$.

(2) When $\mathcal{K} = \mathcal{H}$. By Lemma 1, if the $\mathcal{H}^{**}$-class H_a^{**} contains an idempotent e, then $a\mathcal{H}ae$. Hence $a\mathcal{J}ae$. It follows from the proof of (1) that $ea = a = ae$. Similar to the rest of proof of (1), we can obtain that $\mathcal{H}^{**}$ is a semilattice congruence on S, $S / \mathcal{H}^{**} \cong E(S)$, and each $\mathcal{H}^{**}$-class is a monoid. □

Remark. By the proof of Lemma 3, we have that the $\mathcal{K}^{**}$-class which contains a central idempotent e is a monoid with identity e, and the $\mathcal{K}^{**}$-class contains a unique idempotent.

To characterize the semigroups which are semilattices of $\mathcal{J}^{**}$-simple monoids or $\mathcal{H}^{**}$-simple monoids, we need the following definitions.

A semigroup S is called $\mathcal{K}^{**}$*-simple* if $\mathcal{K}^{**} = S \times S$. A semigroup S is called $\mathcal{K}$*-left cancellative* if for all $a, x, y \in S$,

$$ax \mathcal{K} ay \text{ if and only if } x \mathcal{K} y.$$

As the proof of Lemma 4 in [6], we can obtain the following:

Lemma 4. *A monoid is* $\mathcal{K}^{**}$*-simple if and only if it is* $\mathcal{K}$*-left cancellative.*

Before we give structure theorems for the semigroups in which we are interested, we recall the following notion:

Let Y be a semilattice and for each $\alpha \in Y$, let S_α be a monoid with identity e_α. Assume that S_α are pairwise disjoint. For each $\alpha \geq \beta$ in Y, let $\varphi_{\alpha,\beta}$ be a monoid homomorphism of S_α into S_β such that

(i) $\varphi_{\alpha,\alpha}$ is the identity mapping on S_α for each $\alpha \in Y$;

(ii) $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ if $\alpha \geq \beta \geq \gamma$ in Y.

On $S = \bigcup_{\alpha\in Y} S_\alpha$, define a multiplication $*$ by

$$a * b = (a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta}) \quad (a \in S_\alpha, b \in S_\beta).$$

Then $(S, *)$ is a semigroup which is called a *strong semilattice of the monoids* S_α, $\alpha \in Y$, and is denoted by $[Y; S_\alpha, \varphi_{\alpha,\beta}]$ (see Section III. 7 of [4]).

Lemma 5. *Let* $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ *be a strong semilattice of the monoids* S_α, $\alpha \in Y$. *If* $a \in S_\alpha$, $b \in S_\beta$, *and* $a \mathcal{K} b$, *then* $\alpha = \beta$.

Proof. (1) When $\mathcal{K} = \mathcal{J}$. Since $a\mathcal{J}b$, there exists $x, y \in S^1$ such that $a = xby$. Consider the following three cases:

(i) When x and y are all 1, then $a = b$, and so $\alpha = \beta$. Of course, $\alpha\beta = \alpha$.

(ii) When one of x and y is 1, without loss of generality assume that $y = 1$. Then $a = xb$. Let $x \in S_{\alpha_1}$. Then $a \in S_{\alpha_1\beta}$, hence $\alpha = \alpha_1\beta$. Thus

$$\alpha\beta = \alpha_1\beta \cdot \beta = \alpha_1\beta = \alpha.$$

(iii) When x and y are not 1, that is, $x, y \in S$, assume that $x \in S_{\alpha_1}$, $y \in S_{\beta_1}$. Then $a \in S_{\alpha_1\beta\beta_1}$, and so $\alpha = \alpha_1\beta\beta_1 = \alpha_1\beta_1\beta$. Thus $\alpha\beta = \alpha$.

By the previous discussion, we obtain that $\alpha\beta = \alpha$. From $a\mathcal{J}b$, there exists $u, v \in S^1$ such that $b = uav$. We can similarly obtain that $\beta\alpha = \beta$. But $\alpha\beta = \beta\alpha$. Therefore $\alpha = \beta$.

(2) When $\mathcal{K} = \mathcal{H}$. Since $a\mathcal{H}b$, $a\mathcal{J}b$. By (1), $\alpha = \beta$. $\square$

As the proof of Theorem 6 in [6], we can obtain the following theorem, which gives a structure theorem for the class of all semigroups where all idempotents are central and each $\mathcal{K}^{**}$-class contains an idempotent.

Theorem 6. *For a semigroup S in which* $\mathcal{K}$ *is a left congruence on S, the following are equivalent*:

(1) *the idempotents of S are central, each* $\mathcal{K}^{**}$*-class of S contains an idempotent, and for any* a, b *in each* $\mathcal{K}^{**}$*-class of S, a and b have* $\mathcal{K}$*-relation in the* $\mathcal{K}^{**}$*-class whenever a and b have* $\mathcal{K}$*-relation in S.*

(2) *S is a strong semilattice of* $\mathcal{K}^{**}$*-simple monoids with a unique idempotent, and for any* a, b *in each* $\mathcal{K}^{**}$*-simple monoids, a and b have* $\mathcal{K}$*-relation in the* $\mathcal{K}^{**}$*-simple monoids whenever a and b have* $\mathcal{K}$*-relation in S.*

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